

A Reconstruction Algorithm for a Semilinear Parabolic Inverse Problem

Wuqing Ning, Xue Qin and Yunxia Shang

School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, China

Email: wqning@ustc.edu.cn, qx3@mail.ustc.edu.cn, yxshang@mail.ustc.edu.cn

Abstract: In this paper, we consider an inverse problem to determine a semilinear term of a parabolic equation from a single boundary measurement of Neumann type. For this problem, a reconstruction algorithm is established by the spectral representation for the fundamental solution of heat equation with homogeneous Neumann boundary condition and the Whitney extension theorem.

Keywords: Semilinear parabolic equation, inverse problem, reconstruction algorithm, Whitney extension theorem.

1 Introduction

In this paper, we consider the following initial-boundary value problem

$$\begin{cases} \partial_t u - \Delta u + f(u) = 0 & \text{in } \Omega \times (0, T) \\ u(\cdot, 0) = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \times (0, T) \end{cases} \quad (1)$$

where $\Omega \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) is a C^∞ bounded connected domain, $f \in C^1(\mathbb{R})$ is a non-decreasing function satisfying $f \geq 0$, $f(0) = 0$, and $\varphi \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\partial\Omega \times [0, T])$ ($0 < \alpha < 1$) is such that $\varphi \geq 0$, $\varphi(\cdot, 0) = 0$, $\varphi \not\equiv 0$.

The semilinear parabolic equation in (1) appears, for example, in modeling enzyme kinetics [14] and in other models [1, 19]. Under the above conditions, it can be proved that problem (1) has a unique solution $u = u_f$ which is in $C^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{\Omega} \times [0, T])$ [10, 16], where we denote the solution by $u_f(x, t)$ for specifying the dependence on the semilinear term $f(\cdot)$. In this paper, we are concerned with the following

Inverse Problem. Determine the semilinear term $f = f(\cdot)$ from the Neumann boundary data $\partial_\nu u_f|_{\partial\Omega \times [0, T]}$.

Here ∂_ν denotes the derivative in the direction of the unit outward normal vector to $\partial\Omega$. We should note that, for the inverse problem, only boundary data are known and in $\Omega \times (0, T)$ the solution of problem (1) is still unknown.

Up to now, there have been many research papers on such kind of inverse problem. For uniqueness results under variant additional assumptions, we refer to [2, 5, 7, 15, 17]. For existence results, see for instance [20]. For stability results, we refer to [3, 4] where the domain is a rectangle in [3] and a general smooth domain is considered in [4]. The main ingredients in the proof of the results in [3, 4] are the maximum principle and the gaussian lower bound for the fundamental solution of a certain parabolic operator with Neumann boundary condition.

However, from the point of view of practical applications, it is more important to give theoretically and numerically reconstruction formula or scheme. To the authors' knowledge, there is no result about exact reconstruction formula for f . Therefore, the main object of this paper is trying to solve this problem, and the result is stated as follows.

Theorem 1. *For the inverse problem, it is possible to reconstruct f from φ and $\partial_\nu u_f$ on $\partial\Omega \times [0, T]$.*

The proof of Theorem 1 is based on the spectral representation for the fundamental solution of heat equation with homogeneous Neumann boundary condition and the Whitney extension theorem [21] (see e.g. [8, 9] for the recent development). The reconstruction algorithm established here may be helpful to provide theoretical direction for numerical simulation.

2 Preliminaries

In order to prove Theorem 1, we show some basic facts as follows.

First, it is well known that the initial-boundary value problem

$$\begin{cases} \partial_t v - \Delta v = 0 & \text{in } \Omega \times (0, T) \\ v(\cdot, 0) = 0 & \text{in } \Omega \\ v = \varphi & \text{on } \partial\Omega \times (0, T) \end{cases} \quad (2)$$

has a unique classical solution $v = v_\varphi(x, t)$.

Now let us recall the definition of the fundamental solution of the heat equation with homogeneous Neumann boundary condition. Fix $s \in (s_0, t_0)$. Let $U(x, t; y, s)$ be a continuous function in the region: $s_0 < s < t < t_0$, $x \in \bar{\Omega}$, $y \in \bar{\Omega}$. Then $U(x, t; y, s)$ is called the fundamental solution of the heat equation with homogeneous Neumann boundary condition (also called the heat kernel) if, for any $u_0(x) \in C(\bar{\Omega})$, the function defined by

$$u(x, t) = \int_{\Omega} U(x, t; y, s) u_0(y) dy$$

is a solution of the following initial-boundary value problem

$$\begin{cases} \partial_t u(x, t) - \Delta u(x, t) = 0 & \text{in } \Omega \times (s, t_0) \\ \lim_{t \rightarrow s} u(x, t) = u_0(x) & \text{in } \Omega \\ \partial_\nu u(x, t) = 0 & \text{on } \partial\Omega \times (s, t_0). \end{cases} \quad (3)$$

In fact, $U(x, t; y, s)$ is the unique solution (see e.g. [12]) of the following initial-boundary value problem

$$\begin{cases} \partial_t U(x, t; y, s) - \Delta_x U(x, t; y, s) = 0 & \text{in } \Omega \times (s, t_0) \\ \lim_{t \rightarrow s} U(x, t; y, s) = \delta(x - y) & \text{in } \Omega \\ \partial_\nu U(x, t; y, s) = 0 & \text{on } \partial\Omega \times (s, t_0) \end{cases} \quad (4)$$

where $\delta(\cdot)$ denotes the Dirac delta function.

Next we will give the spectral representation of the fundamental solution. Consider the following eigenvalue problem

$$\begin{cases} -\Delta \omega = \lambda \omega & \text{in } \Omega \\ \partial_\nu \omega = 0 & \text{on } \partial\Omega \end{cases} \quad (5)$$

where λ is parameter. As is well known, the collection Λ of the eigenvalues of problem (5) is real and countable, and $\Lambda = \{\lambda_k\}_{k=1}^\infty$ if we repeat each eigenvalue according to its finite multiplicity as follows

$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \lambda_4 \cdots \rightarrow +\infty.$$

Moreover, there exists an orthonormal basis $\{\omega_k\}_{k=1}^\infty$ of the Hilbert space $L^2(\Omega)$, where ω_k is an eigenfunction corresponding to λ_k :

$$\begin{cases} -\Delta \omega_k = \lambda_k \omega_k & \text{in } \Omega \\ \partial_\nu \omega_k = 0 & \text{on } \partial\Omega. \end{cases} \quad (6)$$

It can be proved that there is an important relationship between the fundamental solution (heat kernel) and the spectrum (see e.g. [6]), that is,

$$U(x, t; y, s) = \sum_{k=1}^{\infty} e^{-\lambda_k(t-s)} \omega_k(x) \omega_k(y) \quad (7)$$

where the convergence is uniform in the region: $t - s \geq \varepsilon$, $x \in \overline{\Omega}$, $y \in \overline{\Omega}$ for arbitrary $\varepsilon > 0$.

3 Proof of Theorem 1

We divide the proof of Theorem 1 into two steps.

Step 1. First we define an admissible set of unknown semilinear terms by

$$\mathcal{F} = \{f \in C^1(\mathbb{R}) : f \geq 0, f(0) = 0, f \text{ is nondecreasing}\}.$$

As is mentioned before, for any $f \in \mathcal{F}$, problem (1) has a unique solution $u_f \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{\Omega} \times [0, T])$. Now let $w = w(x, t) = u_f(x, t) - v_\varphi(x, t)$ where $v_\varphi(x, t)$ is the solution of problem (2). Then by a simple computation we find that w is the unique solution of

$$\begin{cases} \partial_t w - \Delta w = -f(u_f) & \text{in } \Omega \times (0, T) \\ w(\cdot, 0) = 0 & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \times (0, T). \end{cases} \quad (8)$$

In order to express explicitly the solutions of (2) and (8) in terms of the fundamental solution $U(x, t; y, s)$, we replace artificially the corresponding boundary conditions by $\partial_\nu v = \partial_\nu v_\varphi$ and $\partial_\nu w = \partial_\nu u_f - \partial_\nu v_\varphi$ respectively, which can be done because of the solvability of (1) and (2). Then it follows easily from Theorem 9.1 of [12] (take $\alpha = \beta = 0$) that v_φ and w can be expressed by

$$v_\varphi(x, t) = \int_0^t \int_{\partial\Omega} U(x, t; y, s) \partial_\nu v_\varphi(y, s) dS(y) ds \quad (9)$$

and

$$\begin{aligned} w(x, t) &= - \int_0^t \int_{\Omega} U(x, t; y, s) f(u_f(y, s)) dy ds \\ &\quad + \int_0^t \int_{\partial\Omega} U(x, t; y, s) [\partial_\nu u_f - \partial_\nu v_\varphi](y, s) dS(y) ds \end{aligned} \quad (10)$$

for $x \in \overline{\Omega}$ and $t \in [0, T]$, where $dS(y)$ is the surface element on $\partial\Omega$. Here we have made use of the fact that $U(x, t; y, s)$ satisfies the homogeneous Neumann boundary condition in (y, s) (see [12] pp.59). Noting that $v = \varphi$ and $w = 0$ on $\partial\Omega \times [0, T]$, we obtain by (9) and (10) that, for $(x, t) \in \partial\Omega \times [0, T]$,

$$\begin{aligned} I &:= \int_0^t \int_{\Omega} U(x, t; y, s) f(u_f(y, s)) dy ds \\ &= \int_0^t \int_{\partial\Omega} U(x, t; y, s) [\partial_\nu u_f - \partial_\nu v_\varphi](y, s) dS(y) ds \\ &= \int_0^t \int_{\partial\Omega} U(x, t; y, s) \partial_\nu u_f(y, s) dS(y) ds - \varphi(x, t) \\ &:= a(x, t). \end{aligned} \quad (11)$$

Since φ and $\partial_\nu u_f$ on $\partial\Omega \times [0, T]$ are already known, so is $a(x, t)$.

Step 2. In this step we reconstruct $f(\cdot)$. For fixed $s \in [0, T]$, from Section 2 we may expand $f(u_f(\cdot, s))$ in $L^2(\Omega)$ by

$$f(u_f(y, s)) = \sum_{k=1}^{\infty} c_k(s) \omega_k(y). \quad (12)$$

Consequently, in view of (7), for $x \in \overline{\Omega}$, we have

$$I = \sum_{k=1}^{\infty} p_k(t) \omega_k(x), \quad p_k(t) = \int_0^t e^{-\lambda_k(t-s)} c_k(s) ds. \quad (13)$$

On the other hand, by the standard Whitney extension theorem (see e.g. [21]), for fixed $t \in [0, T]$ there exists an extension $\tilde{a}(\cdot, t)$ of the function $a(\cdot, t)$ from $\partial\Omega$ to \mathbb{R}^n . Set $a_k(t) = (\tilde{a}(\cdot, t), \omega_k(\cdot))_{L^2(\Omega)}$ where $(\cdot, \cdot)_{L^2(\Omega)}$ denotes the inner product in $L^2(\Omega)$. In view of (11) and (13), we may expand formally a in the same way. Therefore, if we let $p_k(t) = a_k(t)$, then by (13) it is easy to see that $c_k(\cdot) = (a'_k(\cdot) + \lambda_k a_k(\cdot))$. Hence by (12) we have

$$f(u_f(y, s)) = \sum_{k=1}^{\infty} (a'_k(s) + \lambda_k a_k(s)) \omega_k(y). \quad (14)$$

For any $x \in \partial\Omega$, there exists a sequence $\{y_m\}_{m=1}^{\infty}$ such that $\Omega \ni y_m \rightarrow x$ as $m \rightarrow \infty$. Passing to the limit and Noting $u_f = \varphi$ on $\partial\Omega \times [0, T]$, we obtain by (14)

$$f(\varphi(x, s)) = \sum_{k=1}^{\infty} (a'_k(s) + \lambda_k a_k(s)) \omega_k(x) \quad \text{on } \partial\Omega \times [0, T]. \quad (15)$$

The right hand side of (15) is determined by a and hence by φ and $\partial_v u_f$ on $\partial\Omega \times [0, T]$. And if we let x and s vary on $\partial\Omega \times [0, T]$, then the values of φ vary on the interval $[0, \max_{\partial\Omega \times [0, T]} \varphi]$. Thus by using the known data we can establish a map from $[0, \max_{\partial\Omega \times [0, T]} \varphi]$ to \mathbb{R} , which gives the desired function $f(\cdot)$. \square

Remark. We point out here that the function f reconstructed above is independent of the extension ways of the function a . In fact, this assertion can be proved by the previous results on uniqueness and stability for the inverse problem (see Section 1).

4 Conclusion

In this paper, we have established a reconstruction algorithm for an inverse problem to determine a semilinear term of a parabolic equation from an additional single boundary measurement. The key idea is to find an intrinsic relation between the known data and the unknown function. Although the algorithm formula can not be explicitly written out, in theory it is still important for solving practical problem. The strategy used here may be effective for other similar semilinear inverse problems or even for more challenging nonlinear inverse problems (see e.g. [11, 13, 18]).

Acknowledgement

This work was partially supported by the National Natural Science Foundation of China (Grant No. 11101390), the fundamental Research Funds for the Central University and JSPS Fellowship P05297.

References

- [1] W.F. Ames, *Nonlinear Partial Differential Equations in Engineering* (Academic Press, New York, 1965).
- [2] J.R. Cannon and P. DuChateau, *Structural identification of an unknown source term in a heat equation*, Inverse Problems **14**, 535–551 (1998).
- [3] M. Choulli and M. Yamamoto, *Stable identification of a semilinear term in a parabolic equation*, UTMS (2004), <http://kyokan.ms.u-tokyo.ac.jp/users/preprint/pdf/2004-16.pdf>.
- [4] M. Choulli, E.M. Ouhabaz and M. Yamamoto, *Stable determination of a semilinear term in a parabolic equation*, Commun. Pure Appl. Anal. **5**, 447–462 (2006).
- [5] M. Choulli, *Une introduction aux problèmes inverses elliptiques et paraboliques* (French) (Springer-Verlag, Berlin, 2009).
- [6] J. Dodziuk, *Eigenvalues of the Laplacian and the Heat Equation*, Amer. Math. Monthly **88**, 686–695 (1981).
- [7] P. DuChateau and W. Rundell, *Unicity in an inverse problem for an unknown reaction term in a reaction-diffusion equation*, J. Differential Equations **59**, 155–164 (1985).
- [8] C. Fefferman, *A sharp form of Whitney’s extension theorem*, Ann. of Math. (2) **161**, 509–577 (2005).
- [9] C. Fefferman, *Whitney’s extension problem for C^m* , Ann. of Math. (2) **164**, 313–359 (2006).
- [10] D. Henry, *Geometric Theory of Semilinear Parabolic Equations* (Springer-Verlag, Berlin, 1981).
- [11] V. Isakov, *Inverse problems for partial differential equations* (Second edition) (Springer, New York, 2006).
- [12] S. Itô, *Diffusion equations* (American Mathematical Society, Providence, 1992).
- [13] B. Kaltenbacher and M.V. Klibanov, *An inverse problem for a nonlinear parabolic equation with applications in population dynamics and magnetism*, SIAM J. Math. Anal. **39**, 1863–1889 (2008).
- [14] J.P. Kernevez, *Enzyme Mathematics* (North Holland, Amsterdam, 1980).
- [15] M.V. Klibanov, *Global uniqueness of a multidimensional inverse problem for a nonlinear parabolic equation by a Carleman estimate*, Inverse Problems **20**, 1003–1032 (2004).
- [16] O.A. Ladyzenskaja, V.A. Solonnikov and N.N. Uralceva, *Linear and Quasi-linear Equations of Parabolic Type* (American Mathematical Society, Providence, 1968).

- [17] A. Lorenzi, *An inverse problem for a semilinear parabolic equation*, Annali di Mat. Pura ed Appl. **131**, 145–166 (1982).
- [18] W. Ning, *Uniqueness theorem for a nonlinear parabolic inverse problem*, in: The 2nd International Conference on Multimedia Technology, 6156–6159 (2011).
- [19] J. Ockendon, S. Howison, A. Lacey, and A. Movchan, *Applied Partial Differential Equations* (Oxford University Press, Oxford, 1999).
- [20] M.S. Pilant and W. Rundell, *An inverse problem for a nonlinear parabolic equation*, Comm. Partial Differential Equations **11**, 445–457 (1986).
- [21] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. **36**, 63–68 (1934).